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Twistor theory of symplectic manifolds

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Abstract

This article is a contribution to the understanding of the geometry of the twistor space of a symplectic manifold. We consider the bundle $\mathcal{Z}$ with fibre the Siegel domain $Sp(2n, R)/U(n)$ existing over any given symplectic $2n$-manifold M. Then, after recalling the construction of the almost complex structure induced on $\mathcal{Z}$ by a symplectic connection on M, we study and find some specific properties of both. We show a few examples of twistor spaces, develop the interplay with the symplectomorphisms of M, find some results about a natural almost Hermitian structure on $\mathcal{Z}$ and finally prove its $n+1$-holomorphic completeness. We end by proving a vanishing theorem about the Penrose transform.

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Let (M, ω) be a smooth symplectic manifold of dimension $2n$. Then we may consider the bundle

$$\pi : \mathcal{Z}^l \longrightarrow M \tag{0.1}$$

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of all complex structures j on the tangent spaces to M compatible with ω. Having fibre a cell, the bundle becomes interesting if it is seen with a particular and well known almost complex structure, denoted $\mathcal{J}^\nabla$, — with which we start to treat $\mathcal{Z}^l$ by the name of “Twistor Space” of the symplectic manifold M. The almost complex structure is induced by a symplectic connection on the base manifold and its integrability equation has already been studied [13,14,6,21].

The purpose of this article is two-fold. First, we wish to present results on the complex geometric nature of the twistor space and, second, to show some of the applications to the study of symplectic connections. The almost complex structure $\mathcal{J}^\nabla$ is defined in a very peculiar way, in the sense, for example, that no matter which complex structure is given on the base space M the bundle projection is never pseudo-holomorphic. The complex geometry of the twistor space seems, up to a certain degree which we compute, much more difficult to understand than that one of a regular holomorphic fibre bundle. This is particularly true in the study of the Penrose Transform, which we see as a direct image from cohomology of complex analytic sheaves on $\mathcal{Z}^0$ to real C^∞ sheaves on M. We obtain a “vanishing theorem” which is what one would expect if both spaces were complex and π holomorphic (Section 6). We will also show examples of twistor spaces and a compact generalized “omega” twistor space, in Section 3, and give results about a natural Riemannian structure on $\mathcal{Z}^0$ in Section 5 (following a different approach, such structure has been considered in [13]).

Concerning the second purpose of this work, we relate the twistor almost complex structure to the action of the group of symplectomorphisms of M on the space of symplectic connections. In passing by this independent subject, we present our methods in its study — and apply them on the case of translation invariant symplectic connections, finally just to find a known result proved by other means, cf. [7]. Later we give a new criteria to decide when are two given symplectic connections the affine transformation of one another via a given symplectomorphism of M. We may then claim to have found a description of all germs of twistor spaces of a Riemann surface (Sections 2–4). Moreover, since the integrability equation is immediately and always satisfied in case $n = 1$, with the Levi-Civita connection in particular, we describe a new C^∞ sheaf canonically defined on M, arising from a $\mathbb{C}$-analytic sheaf on $\mathcal{Z}^0$. With this we hope to have contributed to future studies in the field of twistors.

1. The structure of twistor space

Throughout the text let $G = Sp(2n, \mathbb{R})$ and $U^l = U(n-l, l)$, i.e. respectively the groups of symplectic and pseudo-unitary transformations of $\mathbb{R}^{2n}$. Also let $\mathfrak{g}$ and $\mathfrak{u}^l$ denote their respective Lie algebras.

Let $\omega = \sum_{i=1}^n \mathrm{d}x^i \wedge \mathrm{d}y^i$. Recall that the complex symmetric space G/U^l consists of all *real* endomorphisms J such that

$$\begin{cases} J^2 = -\mathrm{Id}, \quad \omega = \omega^{1,1} \text{for } J, \\ \omega(\,, J)\,\text{has signature}\,(2n-2l, 2l). \end{cases}$$

It is known that these spaces are biholomorphic to open cells of the flag manifold $Sp(n)/U(n)$ and that the latter lyes on the complex grassmannian of n-planes in $\mathbb{C}^{2n}$ as the space of lagrangians. Moreover, the boundaries of the G/U^l do not give rise to complex structures under the inverse of the map $J \mapsto \sqrt{-1}$-eigenspace of J, cf. [1]. In a word, there are no other complex structures J of $\mathbb{R}^{2n}$ for which ω is type (1,1).

The vector space $T_J\, G/U^l$ identifies with

$$\mathfrak{m}_J = \{A \in \mathfrak{g} : AJ = -JA\} = [\mathfrak{g}, J]$$

so that $\mathfrak{g} = \mathfrak{m}_J + \mathfrak{u}^l$ and the complex structure is given by left multiplication by J. We have

$$[\mathfrak{g}, \mathfrak{m}_J] = \mathfrak{m}_J, \quad [\mathfrak{m}_J, \mathfrak{m}_J] \subset \mathfrak{u}^l, \quad [\mathfrak{u}^l, \mathfrak{u}^l] \subset \mathfrak{u}^l \tag{1.1}$$

as one can easily compute. It is known that all the symmetric spaces G/U^l are biholomorphic to the first one, when $l = 0$, which is the same as the Siegel Domain or Siegel Upper Half Space $\{X + iY \in \mathbb{C}^{\frac{n(n+1)}{2}} : X, Y$ symmetric matrices, Y positive definite$\}$, and hence that they are all Stein manifolds. Up to this moment we have met the $n + 1$ possible connected fibres, according to $0 \le l \le n$, which will correspond to the various twistor spaces of a symplectic manifold.

Now consider the manifold (M, ω) and the twistor bundle (0.1) with fibre G/U^l. When necessary we denote this space by $\mathcal{Z}^l_M$. We have a short exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow T\mathcal{Z}^l \xrightarrow{\mathrm{d}\pi} \pi^*TM \longrightarrow 0 \tag{1.2}$$

of vector bundles over the manifold $\mathcal{Z}^l$. If we let $E = \pi^*TM$ and denote by Φ the canonical section of End E, given by

$$\Phi_j = j,$$

then we may identify $\mathcal{V} = \ker \mathrm{d}\pi$ with $[\mathfrak{sp}(E, \omega), \Phi]$. This is justified by the theory above applied to the fibre $\pi^{-1}(x)$, which is thus a complex manifold, for all $x \in M$.

We use now a symplectic connection ∇ on TM (see Section 2), in order to construct a horizontal distribution $\mathcal{H}^\nabla$ which splits the sequence (1.2). The following result from the theory of twistor spaces is adapted to our situation, so we recall the proof briefly — to introduce notation and for later reference.

Proposition 1.1 ([15]). $\mathcal{H}^\nabla = \{X \in T\mathcal{Z}^l : (\pi^*\nabla)_X\Phi = 0\}$ *is a complement for* $\mathcal{V}$ *in* $T\mathcal{Z}^l$.

Let $F^{\mathrm{s}}(M)$ be the symplectic frames bundle of M, consisting of all linear symplectic isomorphisms

$$p : \mathbb{R}^{2n} \longrightarrow T_xM.$$

This is a principal G bundle over M and ∇ is determined by the $\mathfrak{g}$-valued 1-form α on $F^{\mathrm{s}}(M)$, given by

$$\nabla_X(sv) = s(s^*\alpha)(X)v$$

for any section s of $F^{\mathrm{s}}(M)$, for $X \in \Gamma TM$, $v \in \mathbb{R}^{2n}$, and by $\alpha(\tilde{A}) = A$ where $A \in \mathfrak{g}$ and $\tilde{A}$ is the vector field

$$\tilde{A}_p = \frac{\mathrm{d}}{\mathrm{d}t}_{|0} p \circ \exp tA.$$

ker α is a horizontal distribution on $F^{\mathrm{s}}(M)$. Fixing $J_0 \in G/U^l$, we get a map

$$\pi_1 : F^s(M) \longrightarrow \mathcal{Z}^l \tag{1.3}$$

given by

$$\pi_1(p) = pJ_0p^{-1}.$$

The derivative of π_1 maps the horizontal distribution on $F^{\mathrm{s}}(M)$ onto a horizontal distribution on $\mathcal{Z}^l$. The proposition is proved by showing that this distribution coincides with $\mathcal{H}^\nabla$.

Notice (1.3) is a principal U^l bundle over $\mathcal{Z}^l$ and that E is also associated to π_1, due to reduction. Since Φ is a section of End E corresponding with the constant *equivariant* function $\hat{\Phi}(p) = p^{-1}\Phi_{\pi_1(p)}p = J_0$ on $F^{\mathrm{s}}(M)$ and since $(\pi^*\nabla)\Phi$ corresponds to the 1-form

$$\mathrm{d}J_0 + [\alpha, J_0],$$

one may then complete the proof of the proposition.

Denoting by

$$P : T\mathcal{Z}^l \longrightarrow \mathcal{V}$$

the projection onto $\mathcal{V}$ with kernel $\mathcal{H}^\nabla$, we have the following consequences.

Proposition 1.2 ([15]). (i) $(\pi^*\nabla)\Phi = [P, \Phi] = 2P\Phi$.
(ii) $T\mathcal{Z}^l = F^s(M) \times_{U^l} \mathfrak{m}_{J_0} \oplus \mathbb{R}^{2n}$.

We may now define, preserving the direct sum $\mathcal{H}^\nabla \oplus \mathcal{V}$, the twistor almost complex structure

$$\mathcal{J}^\nabla = (J^h, J^v),$$

on each point $j \in \mathcal{Z}^l$ as follows: since $\mathrm{d}\pi : \mathcal{H}^\nabla \to E$ is an isomorphism, we transport j from E to the horizontal bundle. J^h is thus, essentially, j itself. The vertical part J^v consists of left multiplication with j, just like in the Siegel Domain.

Let $i = \sqrt{-1}$ and let $j^+ = \frac{1}{2}(1 - ij)$, $j^- = \frac{1}{2}(1 + ij)$ be the projections from $TM \otimes \mathbb{C}$ onto, respectively, T^+M and T^-M, for any $j \in \mathcal{Z}^l$. The integrability equation for $\mathcal{J}^\nabla$ follows from the next theorem, a result which we present in greater generality for later convenience. Notice $\mathcal{J}(M)$ is the general twistor space consisting of all complex structures, i.e. the bundle with fibre $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$, and that, of course, precisely in the same lines of the case we have been considering, any linear connection on M defines a twistor almost complex structure on $\mathcal{J}(M)$.

Theorem 1.1 ([15]). *Let Z be an almost complex manifold and $\pi : Z \to M$ be a smooth submersion onto M with fibres which are smoothly varying complex manifolds. Suppose that Z has a horizontal distribution $\mathcal{H}^Z$ which is j-related to the horizontal distribution $\mathcal{H}^\nabla$ of a connection ∇ on TM via a pseudo-holomorphic smooth* fibre preserving *map*

$$j : Z \longrightarrow \mathcal{J}(M).$$

Then integrability of J^Z implies that the torsion T and curvature R of ∇ satisfy

$$j^+T_x(j^-X, j^-Y) = 0, \quad j^+R_x(j^-X, j^-Y)j^- = 0 \tag{1.4}$$

for all $j \in j(Z)$ and $X, Y \in T_xM$. If j is an immersion these conditions are also sufficient.

In the case of $\mathcal{Z}^l$, j is just the inclusion map and Eqs. (1.4) are equivalent to the vanishing of the Weyl part of the curvature of the given symplectic connection (see Section 2, [6] and [1] for definition and references). In this regard, this Weyl tensor plays a role in symplectic geometry identical to that Weyl tensor of a metric connection in the theory of twistor spaces in Riemannian geometry, cf. [4,15].

We remark that a proof of the independency of the Eqs. (1.4) from the signature, which varies with l, is also written down in [1]. Hence, either all the $\mathcal{Z}^l$ — or none — have integrable almost complex structures.

The following is particular to the symplectic framework.

Theorem 1.2. *If $\mathcal{J}^{\nabla^1} = \mathcal{J}^{\nabla^2}$ then $\nabla^1 = \nabla^2$.*

Proof. Let $A = \nabla^2 - \nabla^1$ and define $\underline{A} \in \Gamma S^3T^*M$ by

$$\begin{aligned}\underline{A}(X, Y, Z) &= \omega(A(X)Y, Z)\\ &= \omega(A(Y)X, Z) = \omega(A(Z)Y, X)\end{aligned}$$

for all $X, Y, Z \in TM$.

Now suppose that $\mathcal{J}^{\nabla^1} = \mathcal{J}^{\nabla^2}$ and u is any ∇^1-horizontal vector field of type (1,0). Then $u = u_2 + v$, with v vertical, also a (1,0) vector field, and u_2 a horizontal vector field for ∇^2. Hence

$$\begin{aligned}[v, \Phi] &= \pi^*\nabla^2_u\Phi\\ &= [\pi^*A(\mathrm{d}\pi\, u), \Phi] \in \mathcal{V}^{(1,0)}\end{aligned}$$

(by Propositions 1.1 and 1.2 and noticing that $[J_0, \mathfrak{m}^+_{J_0}] \subset \mathfrak{m}^+_{J_0}$). In the base manifold this translates as

$$[A_x(j^+X), j] \text{ is } (1, 0), \quad \forall j \in \pi^{-1}(x),\ \forall X \in T_xM.$$

Equivalently this means the projection to $\mathcal{V}^c$ is (1,0), or

$$j^-A(j^+X)j^+ = 0, \quad \forall j,\ \forall X.$$

The equality $\underline{A}^{3,0} = 0, \forall j$, follows immediately. This says $\underline{A}$ must take values in the largest G-invariant subspace of symmetric tensors which satisfy

$$\underline{A}(J_0^+ X, J_0^+ Y, \ldots) = 0$$

for some fixed J_0 and all $X, Y, \ldots$. Indeed, since any $j = gJ_0g^{-1}$ for some $g \in G$, we will also have $j^+ = \frac{1}{2}(1 - ij) = gJ_0^+ g^{-1}$ and therefore

$$\begin{aligned} 0 = (g^{-1} \cdot \underline{A})(J_0^+ X, J_0^+ Y, \ldots) &= \underline{A}(gJ_0^+ X, gJ_0^+ Y, \ldots) \\ &= \underline{A}(j^+ gX, j^+ gY, \ldots). \end{aligned}$$

But $S^k(\mathbb{C}^{2n})$ is irreducible under G for all k, so $\underline{A}$ must be 0. $\square$

2. Symplectic connections

Let M, N be two manifolds and $\sigma : M \to N$ a diffeomorphism between them. Let ∇ be a linear connection on M. Recall that we can define another connection on N by

$$(\sigma \cdot \nabla)_X Y = \sigma \cdot (\nabla_{\sigma^{-1} \cdot X} \sigma^{-1} \cdot Y),$$

where $X, Y \in \mathfrak{X}_N$ and where

$$\sigma \cdot Z_y = \mathrm{d}\sigma(Z_{\sigma^{-1}(y)})$$

for any $Z \in \mathfrak{X}_M$, $y \in N$. It is well defined, at least, on paracompact manifolds.

Indeed, from any tensor on M we can define another one on N. Notice as well that $\sigma \cdot fZ = (f \circ \sigma^{-1})\sigma \cdot Z = \sigma \cdot f\sigma \cdot Z$, for all $f \in \mathrm{C}_M^\infty$, so we prove the last statement and check Leibniz rule for $\sigma \cdot \nabla$. Furthermore, the torsion and curvature satisfy

$$T^{\sigma \cdot \nabla} = \sigma \cdot T^\nabla, \quad R^{\sigma \cdot \nabla} = \sigma \cdot R^\nabla, \tag{2.1}$$

since $\sigma \cdot [Z, W] = [\sigma \cdot Z, \sigma \cdot W]$. Obvious composition rules are satisfied and

$$(\sigma \cdot \nabla)_X \omega = \sigma \cdot (\nabla_{\sigma^{-1} \cdot X} \sigma^* \omega) \tag{2.2}$$

for any form ω on N. For instance, let us prove the last formula:

$$\begin{aligned} &\sigma \cdot (\nabla_{\sigma^{-1} \cdot X} \sigma^* \omega)(Y_1, \ldots, Y_q) = (\nabla_{\sigma^{-1} \cdot X} \sigma^* \omega)(\sigma^{-1} \cdot Y_1, \ldots, \sigma^{-1} \cdot Y_q) \\ &\quad = (\sigma^{-1} \cdot X)(\sigma^* \omega(\sigma^{-1} \cdot Y_1, \ldots, \sigma^{-1} \cdot Y_q)) \\ &\quad - \sum_i \sigma^* \omega(\sigma^{-1} \cdot Y_1, \ldots, \nabla_{\sigma^{-1} \cdot X} \sigma^{-1} \cdot Y_i, \ldots, \sigma^{-1} \cdot Y_q) \\ &\quad = \mathrm{d}(\omega_\sigma(Y_1, \ldots, Y_q))(\mathrm{d}\sigma^{-1}(X)) \\ &\quad - \sum \omega(Y_1, \ldots, (\sigma \cdot \nabla)_X Y_i, \ldots, Y_q) = (\sigma \cdot \nabla)_X \omega(Y_1, \ldots, Y_q). \end{aligned}$$

As we said before, $\sigma^{-1} \cdot \omega = \sigma^* \omega$.

Remark. In a marginal outlook, the above may be applied in Chern-Weil theory to find that all characteristic classes on TM, induced from multilinear forms $f^i : \otimes^i \mathfrak{g} \to \mathbb{R}$ or $\mathbb{C}$, say H-invariant where H is some Lie group, are fixed points of cohomology for every diffeomorphism preserving some H-structure of M. The proof goes as follows. Taking any H-connection ∇, assumed to exist, we then have

$$\begin{aligned} f^i(R^{\sigma^{-1}\cdot\nabla}, \ldots, R^{\sigma^{-1}\cdot\nabla}) &= f^i(\mathrm{d}\sigma^{-1}(\sigma^* R^\nabla)\,\mathrm{d}\sigma, \ldots, \mathrm{d}\sigma^{-1}(\sigma^* R^\nabla)\,\mathrm{d}\sigma) \\ &= \sigma^* f^i(R^\nabla, \ldots, R^\nabla). \end{aligned}$$

Hence, by the independence of the induced de Rham cohomology classes from the choice of the connection, the former are fixed points for σ^*. Of course the result is interesting when $Diff(M)$ has many arcwise-connected components.

From now on we are interested in the case where M and N are symplectic manifolds and σ is a symplectomorphism. Recall that a linear connection on (M, ω) is called symplectic if $\nabla\omega = 0$ and if it is torsion free. In such case, by formulae (2.1) and (2.2), we have that $\sigma \cdot \nabla$ is symplectic too. In particular, we have an action

$$Symp(M, \omega) \times \mathcal{A} \longrightarrow \mathcal{A}$$

on the space of symplectic connections, which preserves the subspace of flat connections. $\mathcal{A}$ is never empty.

Theorem 2.1 (P. Tondeur, cf. [5]). *Every symplectic manifold admits a symplectic connection.*

Furthermore, if a Lie group H acts on M by symplectomorphisms and thus on the space of connections, then M has a H-invariant connection if and only if it has a H-invariant symplectic connection.

Notice that any manifold with a torsion free connection and a non-degenerate parallel two-form is necessarily symplectic.

We now show a few recent results from [5–8,20], for they constitute an important part of the theory of symplectic connections. In order to find a smaller subspace of $\mathcal{A}$ it was introduced in [5] a variational principle

$$\int_M R^2\,\omega^n$$

where $R^2 = \underline{R}_{\alpha\beta\gamma\delta}\underline{R}^{\alpha\beta\gamma\delta}$, with $\underline{R}^{\alpha\beta\gamma\delta} = \underline{R}_{\alpha'\beta'\gamma'\delta'}\omega^{\alpha\alpha'}\omega^{\beta\beta'}\omega^{\gamma\gamma'}\omega^{\delta\delta'}$, and where

$$\underline{R}(X, Y, Z, T) = \omega(R^\nabla(X, Y)Z, T)$$

— a tensor in $\wedge^2 T^*M \otimes S^2 T^*M$. This verifies the first Bianchi identity, because $T^\nabla = 0$, and a second Bianchi identity

$$\underset{X,Y,Z}{\circlearrowright\!\!\!\!\bigoplus} (\nabla_X \underline{R})(Y, Z, T, U) = 0.$$

The representation theory on the space of tensors like $\underline{R}$, under the action of $Sp(2n, \mathbb{R})$, has been determined by I. Vaisman in [20]. It is known that the curvature of ∇ has two irreducible components. So we write $\underline{R} = E + W$ where

$$E(X, Y, Z, T) = -\frac{1}{2(n+1)}\{2\omega(X, Y)r(Z, T) + \omega(X, Z)r(Y, T) + \omega(X, T)r(Y, Z) \\ -\omega(Y, Z)r(X, T) - \omega(Y, T)r(X, Z)\}.$$

$r(X, Y) = \mathrm{Tr}\,\{Z \mapsto R^\nabla(X, Z)Y\}$ is the Ricci tensor. W is called the Weyl tensor and the connection is said to be of Ricci type if $W = 0$. This Weyl part of $\underline{R}$ plays a role parallel to that of the Weyl curvature tensor in Riemannian geometry. In our case too, it is 0 in dimension 2. The variational principle yields the field equations

$$\underset{X,Y,Z}{\circlearrowright\!\!\!\!\bigoplus} (\nabla_X r)(Y, Z) = 0, \tag{2.3}$$

having as particular solutions the Ricci type connections.

In [8], we meet a further characterization of ∇r and find the interesting result that, if $(M_i, \omega_i, \nabla_i)$ are two symplectic manifolds together with corresponding symplectic connections and such that the symplectic $\nabla = \nabla_1 + \nabla_2$ over the cartesian product $(M_1 \times M_2, \omega_1 + \omega_2)$ is of Ricci type, then all three connections must be flat.

A result proved in [6] shows that with its standard 2-form $\mathbb{CP}^n$ is of Ricci type — of course the Levi-Civita connection becomes a symplectic connection in the Kählerian framework.

Still for later purposes, we show the following example. Consider $(\mathbb{R}^2, \omega)$ with the usual coordinates $z = x + iy$ and symplectic structure $\omega = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z} = \mathrm{d}x \wedge \mathrm{d}y$, and let

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Proposition 2.1. *Every symplectic connection on* $(\mathbb{R}^2, \omega)$ *is uniquely determined by two functions* $\alpha, \beta \in C^\infty_M(\mathbb{C})$ *satisfying*

$$\nabla_{\partial_z}\partial_z = \alpha\,\partial_z + \beta\,\partial_{\overline{z}} = \overline{\nabla_{\partial_{\overline{z}}}\partial_{\overline{z}}}$$

and

$$\nabla_{\partial_z}\partial_{\overline{z}} = -\overline{\alpha}\,\partial_z - \alpha\,\partial_{\overline{z}} = \nabla_{\partial_{\overline{z}}}\partial_z.$$

The proof is elementary. Indeed, the *real* and torsion free assumptions, together with

$$\frac{i}{2}\alpha = \omega(\nabla_{\partial_z}\partial_z, \partial_{\bar{z}}) = -\omega(\partial_z, \nabla_{\partial_z}\partial_{\bar{z}}),$$

give us the formula. Because sometimes is impossible to *keep* complex, we show the real correspondent of the last proposition. If

$$\nabla_{\partial_x}\partial_x = b\,\partial_x - a\,\partial_y, \quad \nabla_{\partial_y}\partial_y = d\,\partial_x - c\,\partial_y, \tag{2.4}$$

$$\nabla_{\partial_x}\partial_y = c\,\partial_x - b\,\partial_y = \nabla_{\partial_y}\partial_x, \tag{2.5}$$

with $a, b, c, d : \mathbb{R}^2 \to \mathbb{R}$, then

$$\alpha = -\frac{b+d}{4} - i\frac{a+c}{4} \quad \text{and} \quad \beta = \frac{3b-d}{4} - i\frac{3c-a}{4}.$$

2.1. Translation invariant symplectic connections

Here we study connections in $M = \mathbb{R}^m$. Let s denote the global frame

$$s = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right)$$

and let $\nabla^0 = \mathrm{d}$ be the trivial connection: in the example above, $\alpha = \beta = 0$.

Proposition 2.2. (i) *A connection ∇ is flat iff there exists an open cover $\{U_i\}$ of M and a collection of maps $g_i : U_i \to GL(m, \mathbb{R})$ such that*

$$\nabla s = s g_i \mathrm{d} g_i^{-1}. \tag{2.6}$$

(ii) *Let $\sigma \in Diff(\mathbb{R}^m)$ and $\nabla = \sigma \cdot \nabla^0$. Then the map g given by*

$$g \circ \sigma = \mathrm{Jac}\,\sigma \tag{2.7}$$

satisfies Eq. (2.6) globally.

(iii) *Given any map $g : M \to GL(m, \mathbb{R})$, a necessary condition for Eq. (2.7) to have solution in variable σ, corresponding to Schwarz theorem of mixed derivatives, is that the flat connection defined by formula (2.6) is torsion free.*

(iv) *The isotropy subgroup of ∇^0 is $Diff(M)_{\nabla^0} = GL(m, \mathbb{R}) \rtimes \mathbb{R}^m$.*

Proof.

(i) The condition of ∇ being flat is equivalent to the local existence of parallel frames (solution to a system of quasi-linear differential equations of the first order). The result follows by straightforward computations.

(ii) ∇ is flat since $R^\nabla = \sigma \cdot R^{\nabla^0} = 0$. Now, we have that $\sigma \cdot s = sg$ with some $g : \mathbb{R}^m \to GL(m, \mathbb{R})$. Then

$$\nabla_X sg = \sigma \cdot (\nabla^0_{\sigma^{-1} \cdot X} s) = 0$$

for any vector field X. On the other hand,

$$\nabla\, sg = (\nabla s)g + s\mathrm{d}g = s(Ag + \mathrm{d}g)$$

if $\nabla = \nabla^0 + A$. Hence $Ag = -\mathrm{d}g$, which is equivalent to $A = g\mathrm{d}g^{-1}$. Now, for $\sigma = (\sigma_1, \ldots, \sigma_m)$, since

$$\begin{aligned}\sigma \cdot s &= \left(\mathrm{d}\sigma\left(\frac{\partial}{\partial x_1}(\sigma^{-1})\right), \ldots, \mathrm{d}\sigma\left(\frac{\partial}{\partial x_m}(\sigma^{-1})\right)\right)\\ &= \left(\frac{\partial \sigma_1}{\partial x_1}\frac{\partial}{\partial x_1} + \cdots + \frac{\partial \sigma_m}{\partial x_1}\frac{\partial}{\partial x_m}, \ldots, \frac{\partial \sigma_1}{\partial x_m}\frac{\partial}{\partial x_1} + \cdots + \frac{\partial \sigma_m}{\partial x_m}\frac{\partial}{\partial x_m}\right)(\sigma^{-1})\\ &= s\mathrm{Jac}\,\sigma_{|\sigma^{-1}},\end{aligned}$$

we may deduce formula (2.7).

(iii) Again, notice that $T^\nabla = \sigma \cdot T^{\nabla^0} = 0$ is a necessary condition for the existence of a map σ. We just have to check this agrees with Schwarz theorem. On one hand, if $y = \sigma(x)$,

$$\frac{\partial^2 \sigma_j}{\partial x_i \partial x_k} = \frac{\partial}{\partial x_i}(g_{jk} \circ \sigma) = \frac{\partial g_{jk}}{\partial y_l}\frac{\partial y_l}{\partial x_i} = \frac{\partial g_{jk}}{\partial y_l} g_{li}.$$

On the other hand, if (e_k) is the canonical basis,

$$\nabla_{\partial_i}\partial_k = -s\mathrm{d}g\left(\frac{\partial}{\partial x_i}\right)g^{-1}e_k^T = -s\frac{\partial g_{jt}}{\partial x_i}g^{tk}.$$

This is equal to $\nabla_{\partial_k}\partial_i$ iff

$$\left(\frac{\partial g_{jt}}{\partial x_i}g^{tk}\right)g_{k\alpha}g_{i\beta} = \left(\frac{\partial g_{jt}}{\partial x_k}g^{ti}\right)g_{k\alpha}g_{i\beta}$$

or

$$\frac{\partial g_{j\alpha}}{\partial x_i}g_{i\beta} = \frac{\partial g_{j\beta}}{\partial x_k}g_{k\alpha}$$

which is the equation we were looking for.

(iv) If $\sigma \cdot \nabla^0 = \nabla^0$, then $\mathrm{Jac}\,\sigma$ is constant by Eqs. (2.6) and (2.7). Integrating, we find the group of affine transformations. □

Given a connection ∇, notice the system of partial differential equations $\nabla = \sigma \cdot \nabla^0$ in variable σ is second-order nonlinear. However, we have checked that it is a first-order linear in the entries of $\mathrm{Jac}\,\sigma_{|\sigma^{-1}}$ and easily integrated as such. Indeed, we have just proved that solving $\nabla = \sigma \cdot \nabla^0$ is *equivalent* to solving Eq. (2.7).

Supposing solutions σ exist, composing them with any translation $x \mapsto x+v$, $v \in \mathbb{R}^m$, will also give a solution. So we may look for σ such that $\sigma(0)=0$. Also, assuming $g(0)=1$ is not a problem either, as one deduces from the formula in the proposition — it corresponds to a gauge transformation.

We give a simple example just to illustrate the proposition: consider the open set $\mathbb{R}^+ \times \mathbb{R}$ and, in real coordinate functions, cf. (2.5), take the connection $a=c=0$, $d=x$ and $b=-\frac{1}{2x}$. An easy computation shows ∇ is flat. A little extra work to find the group-valued map g, leads then to the problem of finding (σ_1,σ_2) such that

$$\begin{bmatrix} \dfrac{\sqrt{2\sigma_1}}{2}\mathrm{e}^{-\frac{\sigma_2}{\sqrt{2}}} & -\sqrt{\sigma_1}\mathrm{e}^{\frac{\sigma_2}{\sqrt{2}}} \\ \dfrac{1}{2\sqrt{\sigma_1}}\mathrm{e}^{-\frac{\sigma_2}{\sqrt{2}}} & \dfrac{\sqrt{2}}{2\sqrt{\sigma_1}}\mathrm{e}^{\frac{\sigma_2}{\sqrt{2}}} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial\sigma_1}{\partial x} & \dfrac{\partial\sigma_1}{\partial y} \\ \dfrac{\partial\sigma_2}{\partial x} & \dfrac{\partial\sigma_2}{\partial y} \end{bmatrix}.$$

Notice $\{\sigma_1,\sigma_2\} = \det \mathrm{Jac}\,\sigma = 1$. This is the case where the map g takes values in $SL(2) = Sp(2,\mathbb{R})$. In general, if the map is G-valued, then ∇ is a G-connection.

There is a type of connections for which we have found a solution to the problem raised before. Consider a symplectic connection in $\mathbb{R}^{2n}$ which is translation invariant, that is $T_v \cdot \nabla = \nabla$ for all maps $T_v(x) = x+v$, $v \in \mathbb{R}^{2n}$. Letting $\nabla = \nabla^0 + A$ where A is a $\mathfrak{sp}(2n,\mathbb{R})$-valued 1-form, then we must have

$$T_v \cdot (\nabla^0 + A) = \nabla^0 + T_v \cdot A = \nabla^0 + A.$$

Since $\mathrm{d}T_v = \mathrm{Id}$, one does not take long to conclude that $A_{x+v} = A_x$, i.e. A is a constant 1-form. The following theorem has been proved with entirely different methods.

Theorem 2.2 ([7]). *Let ∇ be a flat, translation invariant and symplectic connection on the manifold $\mathbb{R}^{2n}$. Suppose $\nabla = \nabla^0 + A$. Then $A(X)A(Y)=0$ for all vectors X, Y, and with the map*

$$\sigma(x) = x - \tfrac{1}{2}A(x)x$$

we have $\nabla = \sigma \cdot \nabla^0$.

Proof. First we have

$$0 = R^\nabla = \mathrm{d}^{\nabla^0}A + A \wedge A = A \wedge A$$

so that $[A(X), A(Y)] = 0$ for any pair of vector fields. Hence, to see $A(X)A(Y)=0$, we just have to show $A(X)A(X)=0$. Let $X \in \mathbb{R}^{2n}$ be fixed and consider the two-form

$$\alpha(Y,Z) = \omega(A(X)Y, A(X)Z).$$

By the torsion free assumption, $A(X)Y = A(Y)X$, so α also satisfies

$$\begin{aligned}\alpha(Y, Z) &= \omega(A(Y)X, A(Z)X)\\ &= -\omega(A(Y)A(Z)X, X)\end{aligned}$$

and hence, being symmetric, it must vanish — which implies $A(X)A(X) = 0$. This proves the first part of the theorem.

From Proposition 2.2 we have that $A = g\mathrm{d}g^{-1}$ for some global $g \in \mathrm{A}^0(Sp(2n, \mathbb{R}))$. Certainly, in canonical coordinates $(x_1, \ldots, x_{2n})$

$$A = \sum A_i \,\mathrm{d}x_i = \mathrm{d}\left(\sum x_i A_i\right)$$

with constant A_i. Now let $B = \sum x_i A_i = A(x)$. Again, $\mathrm{d}B\, B = B\,\mathrm{d}B$ so if we put

$$g = \mathrm{e}^{-B} = \sum_{m\geq 0} \frac{(-B)^m}{m!}$$

then $g\mathrm{d}g^{-1} = \mathrm{d}B = A$. According to the same Proposition 2.2 we are left to solve the equation

$$\mathrm{e}^{-A(\sigma)} = \mathrm{Jac}\,\sigma$$

or equivalently

$$1 - A(\sigma) = \mathrm{Jac}\,\sigma.$$

In the canonical basis (e_i) of $\mathbb{R}^{2n}$, this is the same as

$$e_i - A(\sigma)e_i = \frac{\partial\sigma}{\partial x_i}.$$

Letting $\sigma(x) = x - \frac{1}{2}A(x)x$ then on one hand we have

$$A(\sigma(x)) = A(x) - \tfrac{1}{2}A(A(x)x) = A(x)$$

and on the other

$$\frac{\partial\sigma}{\partial x_i} = e_i - \tfrac{1}{2}A(e_i)x - \tfrac{1}{2}A(x)e_i = e_i - A(x)e_i$$

so the given map satisfies the differential equation, as we wished. □

We acknowledge the help of [7] in seeing the $A(X)A(Y) = 0$ part, in dimensions $n \geq 2$. Finally, one may easily find the set of non-zero 1-forms A representing flat, translation invariant symplectic connections in $\mathbb{R}^2$, up to a scalar factor. It is in 1-1 correspondence with the *non-empty* curve

$$\left\{[a:b:c:d]\in P^3(\mathbb{R}) : bc-ad=0, b^2-ac=0\right\}\setminus\{pt\},$$

where $pt=[0:0:1:0]$, which we may compactify by adding the trivial connection.

3. Examples

According to [15], a twistor space over a base space M is an almost complex manifold Z together with a submersion $f: Z\to M$ with fibres almost complex submanifolds. For each z in the fibre $Z_x=f^{-1}(x)$ we have an isomorphism

$$\frac{T_zZ}{\mathcal{V}_z}\longrightarrow T_xM$$

where $\mathcal{V}_z=\ker \mathrm{d}f_z=T_zZ_x$. Then, since the vector space $T_zZ/\mathcal{V}_z$ is complex, we can take this complex structure to T_xM in order to construct a map

$$j: Z\longrightarrow \mathcal{J}(M).$$

Of course f is a pseudo-holomorphic map with respect to some structure on M if, and only if, j is constant along the fibres.

If (M,ω) is a symplectic manifold, we shall call Z an "ω-twistor space" if the image of j is in some $\mathcal{Z}^l$. For example, given a symplectic connection ∇ on M, the tautology of the definition of $\mathcal{J}^\nabla$ proves $\mathcal{Z}^l$ to be a true ω-twistor space.

Recall that the Siegel domain and all G/U^l sit holomorphically and separately in a Grassmannian. So we ask for an extension of $\mathcal{J}^\nabla$ to the compact $Sp(n)/U(n)$-bundle of complex, lagrangian n-planes over the real symplectic $2n$-manifold M. Such extension does not exist (unfortunately), although the standard fibre is a complex symmetric space.

Proposition 3.1. *It is not possible to extend* $(\mathcal{Z}^l,\mathcal{J}^\nabla)$ *to a bigger almost complex manifold of the same dimension, which is also a fibre bundle over M.*

Proof. Assuming the extension to an almost complex space Z exists, the theory above yields a continuous map

$$j:\overline{\mathcal{Z}^l}\longrightarrow \mathcal{J}(M)$$

on the closure of $\mathcal{Z}^l$ in Z, because, if z is any point on the boundary of the twistor space, projecting to a point $x\in M$, then T_zZ_x is still a complex vector space — it is the limit of complex vector subspaces inside a complex vector space.

Also by continuity, we have that $\omega=\omega^{1,1}$ for $j(z)$ and the induced inner product $\omega(\,,j(z)\,)$ has the same signature. However, j is the identity in $\mathcal{Z}^l$ so we arrive to a contradiction. □

Regarding a matter of different nature, it seems that the 'non-constant' compact ω-twistor spaces are not easy to construct or describe.

Proposition 3.2. *There are no ω-twistor spaces with compact fibres of* $\dim > 0$ *satisfying the hypothesis of* Theorem 1.1 *and with the map j an embeding.*

Proof. Assuming Z was such a space, then

$$j : Z \longrightarrow \mathcal{Z}^l$$

would be holomorphic when restricted to each fibre. However, any G/U^l is a Stein manifold so its compact analytic submanifolds are points. □

Clearly the proposition avoids the case of any holomorphic submersion $f : Z \to M$, which induces a map j constant along the fibres.

Here we have the promised examples of twistor spaces.

Example 1. Let $M = \mathbb{R}^2$, ω the canonical symplectic form, ∇ any symplectic connection on M — see Proposition 2.1, from which we use the descriptions and notations in what follows. We want to describe $\mathcal{Z}^0_M$ in terms of its $\overline{\partial}$ operator, since $\mathcal{J}^\nabla$ is always integrable. There is a simple way to see this: R^∇ is a two-form, so it is proportional to ω. Then, since $\omega = \omega^{1,1}$ for $j \in \mathcal{Z}^0_M$, we have $R^\nabla(j^- \ , j^- \ ) = 0$, and so we apply Theorem 1.1 to prove the claim. Otherwise, one may recall that the Weyl part of the curvature is always zero in the two dimensional case.

Now suppose $v \in T^{0,1}M = T^-M$ for j. If $v = \frac{\partial}{\partial z}$ then $j \in -\mathcal{Z}^0 = \mathcal{Z}^1$, so we may already assume, up to a scalar,

$$v = \frac{\partial}{\partial \overline{z}} + w\frac{\partial}{\partial z}$$

for some $w \in \mathbb{C}$. The "positive" condition reads $-i\omega(v, \overline{v}) < 0$. Since

$$-i\omega(v, \overline{v}) = \frac{1}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}\left(\frac{\partial}{\partial \overline{z}} + w\frac{\partial}{\partial z}, \frac{\partial}{\partial z} + \overline{w}\frac{\partial}{\partial \overline{z}}\right) = \frac{1}{2}(w\overline{w} - 1), \tag{3.1}$$

we recover[1] the Siegel disk $\mathcal{D} = \{w : |w| < 1\}$. Because TM is C^∞-trivial we have

$$\mathcal{Z}^0 = M \times \mathcal{D} \xrightarrow{\pi} M.$$

Now working together with $T\mathcal{Z}^0 \otimes \mathbb{C}$ let

$$u = \frac{\partial}{\partial \overline{z}} + w\frac{\partial}{\partial z} + \mathcal{P}\frac{\partial}{\partial w} + \mathcal{Q}\frac{\partial}{\partial \overline{w}}$$

---

[1] In [1] it is proved that the map $J \mapsto -i$-eigenspace is holomorphic, from the Siegel domain, with 'left multiplication by J on T_JG/U^l', to the Grassmannian of complex n-planes in $\mathbb{C}^{2n}$.

be a $\mathcal{J}^{\nabla}$-(0,1)-horizontal vector field, thus projecting to $v = \mathrm{d}\pi(u)$ and where w is the fibre variable. Recall the canonical section $\Phi \in \Gamma(\mathrm{End}\,\pi^*TM)$ defined by $\Phi_j = j$. Then

$$\Phi v = -iv$$

where we see v as a (0,1)-section of $(\pi^*TM)^c$. We can compute the function $\mathcal{P}$ solving

$$\left(\pi^*\nabla_u\Phi\right) v = 0. \tag{3.2}$$

On the left hand side we have — recall Proposition 2.1 —

$$\begin{aligned}
(\pi^*\nabla_u\Phi)v &= \pi^*\nabla_u\Phi v - \Phi\,\pi^*\nabla_u v \\
&= -(i+\Phi)\pi^*\nabla_u v \\
&= -(i+\Phi)\left(\nabla_{\mathrm{d}\pi(u)}\frac{\partial}{\partial\overline{z}} + u(w)\frac{\partial}{\partial z} + w\nabla_{\mathrm{d}\pi(u)}\frac{\partial}{\partial z}\right) \\
&= -(i+\Phi)\left(\nabla_{\partial_{\overline{z}}}\partial_{\overline{z}} + w\nabla_{\partial_z}\partial_{\overline{z}} + \mathcal{P}\frac{\partial}{\partial z} + w\nabla_{\partial_{\overline{z}}}\partial_z + w^2\nabla_{\partial_z}\partial_z\right) \\
&= -(i+\Phi)\left(\overline{\alpha}\frac{\partial}{\partial\overline{z}} + \overline{\beta}\frac{\partial}{\partial z} - \overline{\alpha}w\frac{\partial}{\partial z} - \alpha w\frac{\partial}{\partial\overline{z}} + \mathcal{P}\frac{\partial}{\partial z} - \overline{\alpha}w\frac{\partial}{\partial z} - \alpha w\frac{\partial}{\partial\overline{z}}\right. \\
&\quad \left. + w^2\alpha\frac{\partial}{\partial z} + w^2\beta\frac{\partial}{\partial\overline{z}}\right) \\
&= -(i+\Phi)\left((\overline{\beta} - 2\overline{\alpha}w + \mathcal{P} + w^2\alpha)\frac{\partial}{\partial z} + (\overline{\alpha} - 2\alpha w + w^2\beta)\frac{\partial}{\partial\overline{z}}\right).
\end{aligned}$$

Eq. (3.2) says we are in the presence of a (0,1)-vector for j, therefore by colinearity there exists $\lambda \in \mathbb{C}$ such that

$$(\overline{\beta} - 2\overline{\alpha}w + \mathcal{P} + w^2\alpha)\frac{\partial}{\partial z} + (\overline{\alpha} - 2\alpha w + w^2\beta)\frac{\partial}{\partial\overline{z}} = \lambda\left(\frac{\partial}{\partial\overline{z}} + w\frac{\partial}{\partial z}\right).$$

Henceforth

$$\overline{\beta} - 2\overline{\alpha}w + \mathcal{P} + w^2\alpha = \overline{\alpha}w - 2\alpha w^2 + w^3\beta$$

and thus we get a cubic in w with coefficients in $\mathrm{C}^{\infty}_{\mathbb{R}^2}(\mathbb{C})$:

$$\mathcal{P} = -\overline{\beta} + 3\overline{\alpha}w - 3\alpha w^2 + \beta w^3.$$

To find the function $\mathcal{Q}$ one would have to proceed as above but with (1,0)-vector fields.

Proposition 3.3. (i) *$f \in \mathcal{O}_{\mathcal{Z}^0}$ if and only if*

$$\begin{cases} \dfrac{\partial f}{\partial\overline{w}} = 0 \\ \dfrac{\partial f}{\partial\overline{z}} + w\dfrac{\partial f}{\partial z} + \mathcal{P}(z,w)\dfrac{\partial f}{\partial w} = 0. \end{cases}$$

(ii) *Let* $j : \mathbb{R}^2 \to \mathcal{Z}^0$ *be a section, represented in coordinates by the map* $z \mapsto (z, w(z))$. *Then* j *is* $(j, \mathcal{J}^\nabla)$*-holomorphic iff* w *satisfies*

$$\frac{\partial w}{\partial \overline{z}} + w\frac{\partial w}{\partial z} - \mathcal{P}(z, w(z)) = 0.$$

Proof.

(i) According to the footnote, $\partial/\partial\overline{w}$ is a (0,1)-vector field tangent to the fibres of $\mathcal{Z}^0$, hence the first equation. The second is $u(f) = 0$.

(ii) We consider holomorphic functions f on the twistor space, thus satisfying the system in (i), and then claim that j is $(j, \mathcal{J}^\nabla)$-holomorphic iff $f \circ j$ is holomorphic, $\forall f$. This corresponds to

$$\mathrm{d}(f \circ j)\left(\frac{\partial}{\partial \overline{z}} + w(z)\frac{\partial}{\partial z}\right) = 0.$$

Equivalently,

$$\frac{\partial f}{\partial \overline{z}} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial \overline{z}} + \frac{\partial f}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial \overline{z}} + w\frac{\partial f}{\partial z} + w\frac{\partial f}{\partial w}\frac{\partial w}{\partial z} + w\frac{\partial f}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial z} = 0$$

or

$$\left(-\mathcal{P}(z, w) + \frac{\partial w}{\partial \overline{z}} + w\frac{\partial w}{\partial z}\right)\frac{\partial f}{\partial w} = 0.$$

Since there exist sufficient holomorphic functions, we are finished. □

Remarks.

1. By Darboux's theorem, the proposition describes locally the twistor space of any Riemann surface.
2. We give an independent proof of integrability: for the given basis of (0,1)-vector fields, we have that $\left[\frac{\partial}{\partial \overline{w}}, u\right] = \frac{\partial \mathcal{P}}{\partial \overline{w}}\frac{\partial}{\partial w} + \frac{\partial c}{\partial \overline{w}}\frac{\partial}{\partial \overline{w}} = \frac{\partial c}{\partial \overline{w}}\frac{\partial}{\partial \overline{w}}$ is again a (0,1)-tangent vector. (The almost complex structure $\mathcal{J}_2^\nabla = (J^h, -J^v)$ is never integrable, cf. [1,19], so this computation confirms the correct choices in our example.)
3. In the general theory of twistor spaces, a section j is $(j, \mathcal{J}^\nabla)$ holomorphic if and only if it satisfies a well known condition (cf. [18,19]): $\nabla_u v \in \Gamma T^+M, \forall u, v \in \Gamma T^+M$.

Example 1.1. This is the trivial case; recall the connection ∇^0 is symplectic because $M = \mathbb{R}^2$ is Kähler, so assume $\alpha = \beta = 0$. We have the following global chart:

$$\begin{aligned} \mathcal{Z}^0 = M \times \mathcal{D} &\longrightarrow \mathbb{C} \times \mathcal{D} \\ (z, w) &\longmapsto (w\overline{z} - z, w) \end{aligned}$$

(this map is injective if and only if $|w| \neq 1$). Adding a point at infinity on the right hand side and recalling the grassmannian model of the twistor space, the same map composed with $1/w$ gives a chart of $\mathcal{Z}^1$. Curiously, this example is the only one for which the natural fibre chart w is a *globally* holomorphic function. Also, $\mathbb{C} \times \mathcal{D}$ is convex, so we conclude $\mathcal{Z}^0_M, \mathcal{Z}^1_M$ with complex structure $\mathcal{J}^{\nabla^0}$ are both Stein two-manifolds.

One could also try to find the global charts for the flat torus or cylinder.

Example 2. This is the generalisation of Example 1.1. Let $\omega = \frac{i}{2}\sum_{k=1}^n \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k$. We give a description of $\mathcal{Z}^0_{\mathbb{R}^{2n}}$ with complex structure arising from ∇^0.

First notice that for any element j we can find a basis of T^-M with vectors of the kind

$$v_k = \frac{\partial}{\partial \overline{z}_k} + \sum_l w_{kl}\frac{\partial}{\partial z_l},$$

with $k = 1, \dots, n$, $w_{kl} \in \mathbb{C}$. For, if a linear combination of the $\partial/\partial z_l$, only, were in T^-M, then the positive condition would not be satisfied. Now, ω being (1,1) for j implies

$$0 = \omega(v_{k_1}, v_{k_2}) = \frac{i}{2}(w_{k_1k_2} - w_{k_2k_1}).$$

The positive condition is given by

$$\begin{aligned} 0 > -i\omega(v_k, \overline{v}_k) &= \frac{1}{2}\sum_l \mathrm{d}z_l \wedge \mathrm{d}\overline{z}_l \left(\frac{\partial}{\partial \overline{z}_k} + w_{kp}\frac{\partial}{\partial z_p}, \frac{\partial}{\partial z_k} + \overline{w}_{kq}\frac{\partial}{\partial \overline{z}_q}\right) \\ &= \frac{1}{2}\sum_l (-\delta_{kl} + w_{kl}\overline{w}_{kl}) \\ &= \frac{1}{2}\left(-1 + \sum_l |w_{kl}|^2\right) \end{aligned}$$

where repeated indices in p, q have denoted a sum. With respect to the symmetric matrix $W = [w_{kl}]$ this is equivalent to $1 - WW^* > 0$ and so we meet another well known description of the Siegel domain.

Continuing to reason as in Example 1 we find that a function f on the twistor space is holomorphic if, and only if, $v_k(f) = 0$, $\partial f/\partial \overline{w}_{pq} = 0$. So a global chart for $\mathcal{Z}^0_{\mathbb{R}^{2n}}$ is given by the functions

$$\begin{aligned} &f_{pq}(z_1, \dots, z_n, w_{11}, \dots, w_{n-1,n}) = w_{pq}, \\ &f_k(z_1, \dots, z_n, w_{11}, \dots, w_{n-1,n}) = \overline{z}_k w_{kk} - z_k \end{aligned}$$

where $p \leq q$ and $1 \leq k \leq n$.

Example 3. Consider $M = S^2 = \mathbb{R}^2 \cup \{\infty\}$ with its Kähler metric and corresponding Levi-Civita connection, which is thus symplectic. The two-form is $\omega = \frac{i}{2}\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1+|z|^2)^2}$ so, proceeding as in (3.1), we describe the fibres over the open set $\mathbb{R}^2$ with the disk $\mathcal{D}$ again. Following the theory of hermitian manifolds, the connection is type (1,0), i.e. transforms holomorphic sections in (1,0)-forms. Thus ∇ on T^*M is determined by

$$\nabla \mathrm{d}z = \alpha\,\mathrm{d}z \otimes \mathrm{d}z$$

and a conjugate version of this equation, bearing in mind ∇ is real. Solving $\nabla\omega = 0$ leads to

$$\alpha = \frac{2\bar{z}}{1+|z|^2}.$$

Proceeding then exactly as in Example 1 we find: $f \in \mathcal{O}_{\mathcal{Z}^0_{M-\{\infty\}}}$ if and only if

$$\begin{cases} \dfrac{\partial f}{\partial \bar{w}} = 0 \\ \dfrac{\partial f}{\partial \bar{z}} + w\dfrac{\partial f}{\partial z} + \dfrac{2w(w\bar{z}-z)}{1+|z|^2}\dfrac{\partial f}{\partial w} = 0. \end{cases} \tag{3.3}$$

Now let (z_1, w_1) denote coordinates for the twistor space of M minus the other pole. The affine transformation on the base $z_1 = \sigma(z) = 1/z$ is raised to a $\mathcal{J}^\nabla$-holomorphic transformation of the twistor space. w_1 is defined by requiring that

$$(\mathrm{d}\sigma)^c\left(\frac{\partial}{\partial\bar{z}} + w\frac{\partial}{\partial z}\right) = \lambda\left(\frac{\partial}{\partial \bar{z}_1} + w_1\frac{\partial}{\partial z_1}\right)$$

for some $\lambda \in \mathbb{C}$. That is, the real map $\mathrm{d}\sigma$ applies a (0,1)-w-vector into a (0,1)-w_1-vector. We find

$$w_1 = \frac{\bar{z}^2}{z^2}w$$

and $(z, w) \mapsto (z_1, w_1)$ is holomorphic because one verifies by straightforward computations that if a function $f(z_1, w_1)$ satisfies the system (3.3) in variables (z_1, w_1) then

$$f\left(\frac{1}{z}, \frac{\bar{z}^2}{z^2}w\right)$$

also satisfies the linear system in variables (z, w).

We shall see in the next section that this last result is a manifestation of ∇ being σ-invariant. The latter can be deduced by uniqueness of the Levi-Civita connection after verifying σ is an isometry — which is immediate, since $\mathrm{d}z_1 = -\frac{1}{z^2}\mathrm{d}z$ and thus $\sigma \cdot \omega = \omega$.

Example 4. There exist *compact* ω-twistor spaces: Let $\mathbb{T}^2$ be the torus and consider the trivial bundle

$$Z = \mathbb{T}^2 \times S^2 \xrightarrow{\mathrm{pr}_1} \mathbb{T}^2$$

with almost complex structure J^Z given by the following basis of (0,1)-tangents: the vectors

$$\frac{\partial}{\partial \bar{z}} + \frac{|t|}{1+|t|^2}\frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial \bar{t}}.$$

z is the usual chart of $\mathbb{R}^2$ and t is a fixed affine coordinate of $S^2 = \mathbb{P}^1(\mathbb{C})$. Note that, for $t \neq 0$, we have

$$\frac{|1/t|}{1+|1/t|^2} = \frac{|t|}{1+|t|^2} < 1,$$

so J^Z is well defined and preserves the natural splitting of TZ. Moreover, it is compatible with the canonical symplectic structure of Z. Notice J^Z is not integrable, but this is not important for our purposes.

Hence Z is an ω-twistor space. The map $j : Z \to \mathcal{Z}^0_{\mathbb{T}^2} = \mathbb{T}^2 \times \mathcal{D}$ induced by dpr_1 and the $\mathbb{C}$-vector bundle $TZ/\ker\, \mathrm{dpr}_1$ identifies with

$$j(z,t) = \left(z, \frac{|t|}{1+|t|^2}\right)$$

by construction. For the reader to compare with Proposition 3.2, note that j is not even open along the fibers.

4. A holomorphic map

Let (M,ω), (M_1,ω_1) be two symplectic manifolds and $\sigma : M \to M_1$ a symplectomorphism. Then σ induces an invertible transformation from $\mathcal{Z}^l_M$ onto $\mathcal{Z}^l_{M_1}$ preserving the fibres, i.e. a map Σ such that the diagram

$$\begin{array}{ccc} \mathcal{Z}^l_M & \xrightarrow{\Sigma} & \mathcal{Z}^l_{M_1} \\ \pi \downarrow & & \downarrow \pi_1 \\ M & \xrightarrow{\sigma} & M_1 \end{array}$$

commutes. Indeed, for any $y \in M_1$, $j \in \pi^{-1}(\sigma^{-1}(y))$ we define

$$\Sigma(j) = \mathrm{d}\sigma \circ j \circ \mathrm{d}\sigma^{-1}$$

an element in $\pi_1^{-1}(y)$. It is trivial to check Σ is well defined and invertible.

Assume $\mathcal{Z}^l_M, \mathcal{Z}^l_{M_1}$ have twistor almost complex structures $\mathcal{J}^\nabla$ and $\mathcal{J}^{\nabla^1}$, respectively, where $\nabla^1 = \sigma \cdot \nabla$ and ∇ is a given symplectic connection. We then have the following result.

Theorem 4.1. *Σ is pseudo-holomorphic.*

Proof. Notice that Σ, when restricted to each fibre, extends to a linear map between $\mathrm{End}\, T_{\sigma^{-1}(y)}M$ and $\mathrm{End}\, T_yM_1$. Hence

$$\mathrm{d}\Sigma(jA) = \Sigma(jA) = \Sigma(j)\Sigma(A) = \Sigma(j)\,\mathrm{d}\Sigma(A)$$

and we may conclude the map is *vertically* pseudo-holomorphic.

Now suppose $\Sigma_*\mathcal{H}^\nabla = \mathcal{H}^{\nabla^1}$. Using the isomorphism $\mathrm{d}\pi_1 : \mathcal{H}^{\nabla^1} \to \pi_1^* TM_1$, we have

$$\begin{aligned}\mathrm{d}\pi_1 \circ \mathrm{d}\Sigma\ J^h_j &= \mathrm{d}\sigma \circ \mathrm{d}\pi((\mathrm{d}\pi)^{-1}\, j\, \mathrm{d}\pi) = \mathrm{d}\sigma \circ j \circ (\mathrm{d}\sigma^{-1} \circ \mathrm{d}\sigma) \circ \mathrm{d}\pi \\ &= \Sigma(j)\, \mathrm{d}(\sigma \circ \pi) = \Sigma(j)\mathrm{d}\pi_1 \circ \mathrm{d}\Sigma = \mathrm{d}\pi_1 J^h_{\Sigma(j)}\mathrm{d}\Sigma.\end{aligned}$$

So the theorem follows after we prove $\Sigma_*\mathcal{H}^\nabla = \mathcal{H}^{\nabla^1}$, which is exactly the case when we consider the particular connection ∇^1.

Fix a real symplectic vector space V and let F, F_1 be, respectively, the frame bundles of M and M_1. Consider the G-equivariant map

$$\begin{aligned}\Lambda : F &\longrightarrow F_1 \\ p &\longmapsto \mathrm{d}\sigma \circ p\end{aligned}$$

where the points $p : V \to T_xM$ are linear isomorphisms. If $s : U \to F$ is a section on a neighborhood U of $x \in M$, then

$$s_1 = \Lambda \circ s \circ \sigma^{-1} : \sigma(U) \longrightarrow F_1$$

is a section on a neighborhood of $\sigma(x)$. We wish to show first that Λ preserves the horizontal distributions induced by the connections. Let α, α_1 denote the connection 1-forms on F and F_1.

$$\nabla_{X_x} s = s(s^*\alpha)(X_x)$$

and

$$\begin{aligned}(\sigma \cdot \nabla)_{Y_{\sigma(x)}} s_1 &= s_1(s_1^*\alpha_1)(Y_{\sigma(x)}) = \Lambda \circ s \circ \sigma^{-1}_{\sigma(x)}[(\Lambda \circ s \circ \sigma^{-1})^*\alpha_1](Y_{\sigma(x)}) \\ &= \mathrm{d}\sigma\, s(s^*\Lambda^*\alpha_1)\mathrm{d}\sigma^{-1}(Y_{\sigma(x)}) = \mathrm{d}\sigma\, s(s^*\Lambda^*\alpha_1)(\sigma^{-1} \cdot Y)_x.\end{aligned}$$

On the other hand, since $(\sigma^{-1} \cdot s_1)_x = \mathrm{d}\sigma^{-1}(s_{1\sigma(x)}) = s_x$, we have

$$(\sigma \cdot \nabla)_{Y_{\sigma(x)}} s_1 = \sigma \cdot (\nabla_{\sigma^{-1}\cdot Y}\, \sigma^{-1} \cdot s_1)_{\sigma(x)} = \mathrm{d}\sigma(\nabla_{(\sigma^{-1}\cdot Y)_x}\, s) = \mathrm{d}\sigma\, s(s^*\alpha)(\sigma^{-1} \cdot Y)_x.$$

Henceforth $s^*\Lambda^*\alpha_1 = s^*\alpha$ and we prove the claim that $\ker\alpha_1 = \Lambda_*\ker\alpha$ by taking horizontal frames along paths in M passing through x. (With vertical fundamental vector fields one can actually see further that $\Lambda^*\alpha_1 = \alpha$.)

Finally let $\zeta : F \to Z$ be the once introduced fibre bundle (cf. first section, formula (1.3)) with bundle map

$$\zeta(p) = pJ_0p^{-1},$$

where $J_0 \in G/U^l$ is some compatible complex structure of V. Clearly

$$\Sigma\circ\zeta(p) = \mathrm{d}\sigma\, pJ_0p^{-1}\mathrm{d}\sigma^{-1} = \zeta_1\circ\Lambda(p)$$

and we know the ζ preserve the horizontal tangent bundles:

$$\zeta_*\ker\alpha = \mathcal{H}^\nabla,\ \zeta_{1*}\ker\alpha_1 = \mathcal{H}^{\sigma\cdot\nabla}.$$

Now it is no longer difficult to see that $\Sigma_*\mathcal{H}^\nabla = \mathcal{H}^{\sigma\cdot\nabla}$. □

We notice that the construction and results above are true for the general twistor space $\mathcal{J}(M)$. Indeed, the proof does not mention any particular feature of symplectic manifolds.

Remark. An application of the last theorem is the result at the end of Example 3 in Section 3. The theorem confirms that the PDE system given there is preserved under the change of affine coordinates in S^2. It also applys in the following strictly real situation: since $(\mathbb{R}^2, \omega)$ is symplectomorphic to the Poincaré disk $(\mathcal{D}, \omega_1)$, where

$$\omega_1 = \frac{i}{2}\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1-|z|^2)^2},$$

we can study $\mathcal{Z}^l_{\mathcal{D}}$ using the theorem and Example 1 in Section 3 (it corresponds to find the Darboux coordinates in $\mathcal{D}$ and the respective connection's parameters).

There is a partial converse to the theorem, which is only valid in the symplectic category. In the following we assume all the previous setting.

Corollary 4.1. *Let ∇^2 be any symplectic connection on M_1 and suppose $\Sigma : (\mathcal{Z}^l_M, \mathcal{J}^\nabla) \to (\mathcal{Z}^l_{M_1}, \mathcal{J}^{\nabla^2})$ is holomorphic. Then*

$$\nabla^2 = \sigma\cdot\nabla$$

i.e. ∇^2 is in the affine transformation orbit of ∇.

Proof. We have

$$\mathcal{J}^{\nabla^2} = \mathrm{d}\Sigma\circ\mathcal{J}^\nabla\circ\mathrm{d}\Sigma^{-1} = \mathcal{J}^{\sigma\cdot\nabla}$$

so the result follows by Theorem 1.2. □

We remark that the theorem has the apparent merit of transforming a second-order PDE's problem into a first-order one.

5. The metric

In order to introduce a Riemannian structure on the twistor space $\mathcal{Z}_M^0$ we need a further amount of theory from [15]. Recall the exact sequence (1.2), where $E = \pi^*TM$ is a vector bundle over $\mathcal{Z}_M^0$ with canonical complex structure Φ. Also important to recall here are Propositions 1.1 and 1.2.

Let ∇ be a symplectic linear connection on the given $2n$-dimensional symplectic manifold M. Let $P \in \mathrm{A}^1(\mathcal{V})$ denote the projection with kernel $\mathcal{H}^\nabla$, induced by the connection. Via the identity

$$\mathcal{V}_j = \{A \in \mathfrak{sp}(E_j, \pi^{-1}\omega) : A\Phi_j = -\Phi_j A\}$$

P can also be seen as an endomorphism-valued one-form on the twistor space. We may thus define a new connection on E by

$$D = \pi^*\nabla - P,$$

which turns $\pi^*\nabla\Phi = [P, \Phi]$ equivalent to

$$D\Phi = 0.$$

It follows that D on $\mathrm{End}\, E$ preserves $\mathcal{V}$ and hence $DJ^v = 0$. Indeed, this connection is symplectic because its difference to an obviously symplectic connection $\pi^*\nabla$ stays within $\mathfrak{sp}(E, \pi^{-1}\omega)$, and hence, as a derivation, acts trivially on the two-form.

The isomorphism $\pi_* : \mathcal{H}^\nabla \to E$ allows us to transfer D, to give rise to a new connection D on $\mathcal{H}^\nabla$ satisfying

$$\begin{aligned}(DJ^h)X &= \pi_*^{-1}(D(\pi_* J^h X)) - J^h \pi_*^{-1}(D\pi_* X)\\ &= \pi_*^{-1}(D\Phi)\pi_* X = 0.\end{aligned}$$

Henceforth we have defined a $\mathbb{C}$-linear connection on $T\mathcal{Z}^0 = \mathcal{V} \oplus \mathcal{H}^\nabla$ preserving this splitting, exactly in the same lines of the general twistor theory ([15]). Since π_* resulted in a parallel and $\mathbb{C}$-linear isomorphism, one often identifies $\mathcal{H}^\nabla$ with E.

Now we need the following theorem valid in general in $\mathcal{J}(M)$ and which we may improve in a little detail.

Theorem 5.1 ([15]). *The connection D on the tangent bundle of $\mathcal{Z}_M^0$ has torsion whose vertical part is the projection of $\pi^*R^\nabla - \frac{1}{2} P \wedge P$ into $\mathcal{V}$, and whose horizontal part is $\pi^*T^\nabla - P \wedge \mathrm{d}\pi$ after identifying $\mathcal{H}^\nabla$ with E.*

Since $[\mathfrak{m}_J, \mathfrak{m}_J] \subset \mathfrak{gl}(2n, J)$, cf. Section 1, one concludes that the vertical part of T^D is just $P(\pi^* R^\nabla)$. Also notice we are already assuming ∇ is torsion free, so both formulas in the theorem can be simplified.

The present section is devoted to the study of a natural Riemannian structure on $\mathcal{Z}^0_M$, whose analogous construction in 'Riemannian twistor theory' has been already considered in [18]. To see which twistor spaces of that kind over a four-manifold admit a Kähler metric one may consult [10]. For the symplectic case, especially $\mathbb{R}^{2n}$ canonical, one may also consult [13].

Recall that G/U^0 is a Hermitian symmetric space, hence Kählerian. With the help of the Killing form and a Cartan's decomposition of $\mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{u}^0 \oplus \mathfrak{m}_J$ one defines a symplectic form on $\mathcal{Z}^0$ by

$$\Omega^\nabla = t\,\pi^*\omega - \tau,$$

where $t \in]0, +\infty[$ is fixed and

$$\tau(X, Y) = \tfrac{1}{2}\mathrm{Tr}\,(PX)\Phi(PY).$$

The following is trivial to check.

Lemma 5.1. *Ω^∇ is non-degenerate and $\mathcal{J}^\nabla$ is compatible with it. The induced metric is positive definite.*

Although the parameter t will not teach us anything special about the twistor space, besides that it could also give a pseudo-metric, it may become important at some moment.

Proposition 5.1. *For any $X, Y, Z \in T\mathcal{Z}^0$*

$$\mathrm{d}\tau(X, Y, Z) = -\tfrac{1}{4}\mathrm{Tr}\,(R^{\pi^*\nabla}_{X,Y} \circ \pi^*\nabla_Z\Phi + R^{\pi^*\nabla}_{Y,Z} \circ \pi^*\nabla_X\Phi + R^{\pi^*\nabla}_{Z,X} \circ \pi^*\nabla_Y\Phi).$$

Proof. Let us first see $D\Omega^\nabla = 0$. By previous remarks we are left to check $D\tau = 0$.

$$\begin{aligned} D_X\tau\,(Y, Z) &= X(\tau(Y, Z)) - \tau(D_XY, Z) - \tau(Y, D_XZ) \\ &= X(\tau(Y, Z)) - \tfrac{1}{2}\mathrm{Tr}\,(P(D_XY)\Phi PZ + PY\Phi P(D_XZ)) \\ &= X(\tau(Y, Z)) - \tfrac{1}{2}\mathrm{Tr}\,D_X(PY\,\Phi\,PZ) \\ &= X(\tau(Y, Z)) - \mathrm{d}(\tfrac{1}{2}\mathrm{Tr}\,(PY\,\Phi\,PZ))(X) = 0. \end{aligned}$$

Now, it is well known that

$$\mathrm{d}\tau(X, Y, Z) = \tau(T^D_{X,Y}, Z) + \tau(T^D_{Y,Z}, X) + \tau(T^D_{Z,X}, Y).$$

Since

$$\begin{aligned} \tau(T^D_{X,Y}, Z) &= \tfrac{1}{4}\mathrm{Tr}\,([PT^D_{X,Y}, \Phi]PZ) = -\tfrac{1}{4}\mathrm{Tr}\,(\pi^* R^\nabla_{\pi_*X,\pi_*Y}[PZ, \Phi]) \\ &= -\tfrac{1}{4}\mathrm{Tr}\,(R^{\pi^*\nabla}_{X,Y} \circ \pi^*\nabla_Z\Phi), \end{aligned}$$

the result follows. $\square$

Theorem 5.2. *Ω^∇ is closed if and only if ∇ is flat. In such case, $\mathcal{Z}_M^0$ is a Kähler manifold.*

Proof. Since $\mathrm{d}\pi^*\omega = 0$, we only have to do an analysis of $\mathrm{d}\tau$ on four cases — with three horizontal or vertical tangent vectors X, Y, Z.

The only possible non-trivial case is say X, Y horizontal and Z vertical. Then, since τ on $\mathcal{V}$ is non-degenerate, $\mathrm{d}\tau(X, Y, Z) = \tau(T^D_{X,Y}, Z) = 0$ for all those X, Y, Z iff $P(T^D) = 0$. Equivalently, $[\pi^* R^\nabla, \Phi] = 0$, or

$$[R^\nabla_x, j] = 0, \quad \forall j \in \pi^{-1}(x),\ x \in M.$$

Now, for any J compatible with $(\mathbb{R}^{2n}, \omega)$, let $\mathfrak{u}^0_J$ be the unitary Lie algebra $\mathfrak{sp}(2n, \mathbb{R}) \cap \mathfrak{gl}(2n, J)$. It is then trivial to see that

$$\mathfrak{h} = \bigcap_{J \in G/U^0} \mathfrak{u}^0_J$$

is a G-module under the adjoint action. Because $\mathfrak{sp}(2n, \mathbb{R})$ is irreducible, we have $\mathfrak{h} = 0$ and thus the 'only if' part of the theorem.

For the last part of the theorem we recall that $R^\nabla = 0$ implies integrability of the almost complex structure $\mathcal{J}^\nabla$ as well. $\square$

Notice D is always Hermitian, Ω^∇ may be Kählerian, but T^D is never 0. Thus the (0,1) part of D cannot be the $\overline{\partial}$ operator.

Let $\langle , \rangle$ be the induced metric, so that

$$\langle X, Y \rangle = t\,\pi^*\omega(X, \mathcal{J}^\nabla Y) + \tfrac{1}{2}\mathrm{Tr}\,(PXPY)$$

and thus $\mathcal{H}^\nabla \perp \mathcal{V}$. Let $\cdot^v$ denote the vertical part of any tangent-valued tensor.

Theorem 5.3. (i) *The Levi-Civita connection of $\langle\ ,\ \rangle$ is given by*

$$\mathfrak{D}_X Y = D_X Y - PY(\pi_* X) - \tfrac{1}{2}\pi^* R^v_{X,Y} + S(X, Y)$$

where S is symmetric and defined both by

$$\langle S^v(X, Y), A \rangle = \langle A\pi_* X, \pi_* Y \rangle, \quad \forall A \in \mathcal{V},$$

and

$$\langle S^h(X, B), Y \rangle = \tfrac{1}{2}\langle \pi^* R^v_{X,Y}, B \rangle, \quad \forall Y \in \mathcal{H}^\nabla.$$

Hence for $X, Y \in \mathcal{H}^\nabla$ *and* $A, B \in \mathcal{V}$ *we have*

$$S^v(X, A) = S^v(A, B) = 0,$$
$$S^h(X, Y) = S^h(A, B) = 0.$$

(ii) *The fibres* $\pi^{-1}(x)$, $x \in M$, *are totally geodesic in* $\mathcal{Z}^0_M$.
(iii) *If* ∇ *is flat, then* $\mathfrak{D}\mathcal{J}^\nabla = 0$.

Proof. (i) Note that S^h is symmetric by definition and that, to confirm S^v is symmetric, we just have to check every $A \in \mathcal{V}$ is self-adjoint:

$$\begin{aligned}\langle A\pi_*X, \pi_*Y\rangle &= t\,\omega(A\pi_*X, \Phi\pi_*Y)\\ &= t\,\omega(\pi_*X, \Phi A\pi_*Y) = \langle \pi_*X, A\pi_*Y\rangle.\end{aligned}$$

Now let us see the torsion condition:

$$\begin{aligned}T^{\mathfrak{D}}(X, Y) = T^D(X, Y) &- PY(\pi_*X) - \tfrac{1}{2}\pi^*R^v_{X,Y} + S(X, Y) + PX(\pi_*Y)\\ &+\tfrac{1}{2}\pi^*R^v_{Y,X} - S(Y, X)\\ &= T^D(X, Y) + P \wedge \mathrm{d}\pi(X, Y) - \pi^*R^v_{X,Y} = 0.\end{aligned}$$

For the metric condition it is wise, from now on, to let X, Y, Z denote horizontal and A, B, C vertical vector fields. We already know D is Hermitian, so to simplify computations let $\xi = \mathfrak{D} - D$. Then

$$\xi_X Y = -\tfrac{1}{2}\pi^*R^v_{X,Y} + S^v(X, Y), \qquad \xi_X A = -AX + S^h(X, A),$$
$$\xi_A X = S^h(X, A), \qquad \xi_A B = 0$$

and thus in particular, from the last formula, we deduce (ii). Now

$$\mathfrak{D}_X\langle\,,\,\rangle(Y, Z) = -\langle \xi_X Y, Z\rangle - \langle Y, \xi_X Z\rangle = 0,$$

$$\begin{aligned}\mathfrak{D}_X\langle\,,\,\rangle(Y, A) &= -\langle \xi_X Y, A\rangle - \langle Y, \xi_X A\rangle\\ &= \tfrac{1}{2}\langle \pi^*R^v_{X,Y}, A\rangle - \langle S^v(X, Y), A\rangle + \langle Y, AX\rangle - \langle Y, S^h(X, A)\rangle = 0,\end{aligned}$$

$$-\mathfrak{D}_X\langle\,,\,\rangle(A, B) = \langle \xi_X A, B\rangle + \langle A, \xi_X B\rangle = 0,$$

$$\begin{aligned}-\mathfrak{D}_A\langle\,,\,\rangle(X, Y) &= \langle S^h(X, A), Y\rangle + \langle X, S^h(Y, A)\rangle\\ &= \tfrac{1}{2}\langle \pi^*R^v_{X,Y}, A\rangle + \tfrac{1}{2}\langle \pi^*R^v_{Y,X}, A\rangle = 0,\end{aligned}$$

$$-\mathfrak{D}_A\langle\,,\,\rangle(X, B) = \langle \xi_A X, B\rangle + \langle X, \xi_A B\rangle = 0,$$

and finally

$$-\mathfrak{D}_A\langle\,,\,\rangle(B, C) = 0.$$

(iii) We already know this, but we are glad to confirm: if ∇ is flat then $S^h = 0$. Hence for *all* vector fields

$$\mathfrak{D}_X\mathcal{J}^\nabla Y = \mathcal{J}^\nabla D_X Y - \mathcal{J}^\nabla PY(\pi_* X) + S^v(X, \mathcal{J}^\nabla Y).$$

It is an easy task to show $\langle S^v(X, \mathcal{J}^\nabla Y), A\rangle = \langle \mathcal{J}^\nabla S^v(X, Y), A\rangle$ (an identity also following from the theory of the 2nd fundamental form in Kähler geometry), so we are finished. $\square$

One may write S^v explicitly and construct a symplectic-orthonormal basis of $\mathcal{V}$ induced by a given such basis on $\mathcal{H}^\nabla$. We show the first of these assertions.

Proposition 5.2. *For X, Y horizontal*

$$S^v_j(X, Y) = -\tfrac{1}{2}t\{\omega(X,\ )jY + \omega(jY,\ )X + \omega(jX,\ )Y + \omega(Y,\ )jX\}.$$

Proof. Since this formula is clearly symmetric we just have to verify that $S^v(X, X) \in \mathcal{V}$ and $\langle S^v(X, X), A\rangle = \langle AX, X\rangle$ for any vertical vector $A \in \mathcal{V}$. For the first part

$$\begin{aligned}\omega(S^v_{X,X}Y, Z) &= -t\omega(\omega(X, Y)jX + \omega(jX, Y)X, Z)\\ &= -t\{\omega(X, Y)\omega(jX, Z) + \omega(jX, Y)\omega(X, Z)\} = \omega(S^v_{X,X}Z, Y)\end{aligned}$$

and

$$\begin{aligned}S^v_{X,X}j &= -t\{\omega(X, j\ )jX + \omega(jX, j\ )X\}\\ &= tj\{\omega(jX,)X + \omega(X,)jX\} = -jS^v_{X,X}.\end{aligned}$$

Now let $(e_1, \ldots, e_n, je_1, \ldots, je_n)$ be an orthonormal and symplectic basis of $\mathcal{H}^\nabla_j \simeq T_{\pi(j)}M$. Then

$$\begin{aligned}\langle S^v(X, X), A\rangle &= \tfrac{1}{2}\mathrm{Tr}\, S^v(X, X)A\\ &= \tfrac{1}{2}\omega(S^v(X, X)Ae_i, je_i) + \tfrac{1}{2}\omega(S^v(X, X)Aje_i, j^2e_i)\\ &= \omega(S^v(X, X)Ae_i, je_i)\\ &= -t\omega(\omega(X, Ae_i)jX + \omega(jX, Ae_i)X, je_i)\\ &= t\omega(AX, \omega(jX, je_i)e_i) - t\omega(AX, \omega(jX, e_i)je_i)\\ &= t\omega(AX, jX) = \langle AX, X\rangle. \quad \square\end{aligned}$$

5.1. Kählerian twistor spaces

Next we present a result about the sectional curvature of the Kählerian twistor space $\mathcal{Z}^0$. Since the result is not used anymore we do not show its long proof. Until the end of the subsection assume $R^\nabla = 0$.

Theorem. *Let Π be a two-plane in $T_j\mathcal{Z}^0$ spanned by the orthonormal basis $\{X + A, Y + B\}$, $X, Y \in \mathcal{H}^\nabla$, $A, B \in \mathcal{V}$. Then the sectional curvature of Π is*

$$\begin{aligned} k_j(\Pi) &= -\langle R^{\mathfrak{D}}(X + A, Y + B)(X + A), Y + B\rangle \\ &= \tfrac{1}{2}(||X||^2||Y||^2 + 3t^2\omega(X, Y)^2 - \langle X, Y\rangle^2) + ||BX - AY||^2 \\ &\quad +2\langle [A, B]X, Y\rangle - ||[A, B]||^2 \end{aligned}$$

where $[,]$ is the commutator bracket. Thus,

$$k_j(\Pi) \begin{cases} > 0 \text{ for } \Pi \subset \mathcal{H}^\nabla \\ < 0 \text{ for } \Pi \subset \mathcal{V}. \end{cases}$$

We remark that the second part of the theorem can be obtained immediately from Gauss-Codazzi's equations. First, notice that the horizontal distribution is integrable when ∇ is flat. Then the horizontal leaves are immediately seen to have $\pi^*\nabla$ for Levi-Civita connection with the induced metric, and hence they are flat. Finally, for X, Y horizontal and orthonormal, and being S the second fundamental form, a formula of Gauss says

$$\begin{aligned} k_j\{X, Y\} &= ||S(X, Y)||^2 - \langle S(X, X), S(Y, Y)\rangle \\ &= \langle S(X, Y)X, Y\rangle - \langle S(X, X)Y, Y\rangle = \text{etc} \end{aligned}$$

which is positive, as we may deduce following Proposition 5.2. For the totally geodesic vertical fibres of $\mathcal{Z}^0$, we recall that $-||[A, B]||^2$ is the sectional curvature of the hyperbolic space $Sp(2n, \mathbb{R})/U(n)$.

One can find the Cauchy-Riemann operator on the tangent bundle of $\mathcal{Z}^0$. We shall proceed to do this, hoping to bring further understanding to the Kählerian case.

Proposition 5.3. (i) *A tangent vector field Y on $\mathcal{Z}^0_M$ is holomorphic iff*

$$D_X Y + \mathcal{J}^\nabla D_{\mathcal{J}^\nabla X} Y - 2(PY)\pi_* X = 0, \quad \forall X.$$

(ii) $\mathcal{H}^\nabla$ *is a holomorphic subvector bundle of $T\mathcal{Z}^0$.*
(iii) R^D *is a* (1,1)*-form.*

Proof. (i) It is well known that $\overline{\partial} = {}'' \circ \mathfrak{D}$ when we see the tangent space as a $\mathbb{C}$-vector bundle. Hence

$$\begin{aligned}\overline{\partial}_{X+i\mathcal{J}^\nabla X}(Y - i\mathcal{J}^\nabla Y) &= \mathfrak{D}_X Y + \mathfrak{D}_{\mathcal{J}^\nabla X}\mathcal{J}^\nabla Y + i(\mathfrak{D}_{\mathcal{J}^\nabla X} Y - \mathcal{J}^\nabla \mathfrak{D}_X Y)\\ &= \mathfrak{D}_X Y + \mathcal{J}^\nabla \mathfrak{D}_{\mathcal{J}^\nabla X} Y - i\mathcal{J}^\nabla(\mathfrak{D}_X Y + \mathcal{J}^\nabla \mathfrak{D}_{\mathcal{J}^\nabla X} Y).\end{aligned}$$

Therefore, $\overline{\partial}$ operates as the real part of the above, which is equal to

$$\begin{aligned}&D_X Y - (PY)\pi_* X + S(X,Y) + \mathcal{J}^\nabla D_{\mathcal{J}^\nabla X} Y - \mathcal{J}^\nabla (PY)\pi_*\mathcal{J}^\nabla X + \mathcal{J}^\nabla S(\mathcal{J}^\nabla X, Y)\\ &\quad = D_X Y + \mathcal{J}^\nabla D_{\mathcal{J}^\nabla X} Y - 2(PY)\pi_* X.\end{aligned}$$

(ii) We have seen D is a Hermitian connection on $\mathcal{H}^\nabla \simeq E$. From the formula above we immediately find that D determines a $\overline{\partial}$-operator on E coinciding with $\overline{\partial}$, hence integrable. Moreover, by a famous theorem of Koszul-Malgrange ([11]), R^D must not have (0,2)-part.

(iii) This follows from (ii). However, one may argue as in Corollary 5.1, formula (5.2). □

Notice $\mathcal{V} \subset \operatorname{End} E$ also inherits an integrable complex structure as a manifold, by part (ii). However, this has no longer anything to do with $\mathfrak{D}$ or $\mathcal{J}^\nabla$.

In conclusion, the Kählerian twistor space $\mathcal{Z}^0_M$ has holomorphic charts in $\mathbb{C}^n \times \mathbb{C}^{\frac{1}{2}n(n+1)}$ like

$$H \times W \quad \text{or} \quad U \times V$$

with $H \times \{w\}$ horizontal and $\{x\} \times V$ vertical, but never a chart of the kind $H \times V$. This is not new though; it agrees with the fact that the bundle projection π is never holomorphic.

5.2. Twistor space of a Riemann surface

Until the end of this section assume (M, ω, J_0) is a Riemann surface. Then there are various ways to describe $\mathcal{Z}^0_M$. For example, combining the well known isomorphism $j \mapsto (j + J_0)^{-1}(j - J_0)$, valid in general, with an extra property of real dimension two, one may deduce easily that $\mathcal{Z}^0_M$ is diffeomorphic to the radius 1 disk bundle of $T^+M \otimes_c T^+M$ (see [1]). This transformation is particularly suitable for the study of $\mathcal{J}^\nabla$ with ∇ reducible to $U(1)$: we then get a biholomorphism between ω-twistor spaces.

Suppose M is connected, orientable and compact. Then its Euler characteristic is equal to $2 - 2g$ where g is the genus of M. We know a way to embed $\mathcal{Z}^0_M$ in $\mathbb{P}^1(TM \otimes \mathbb{C})$. Since this is associated to an even Euler number bundle, we may use a result from [12] on the classification of sphere bundles over Riemann surfaces to conclude that it is diffeomorphic to the trivial bundle $M \times S^2$. Hence it yields that M parameterizes a disc flowing inside S^2, the twistor's fibres, with boundary *the* principal $U(1)$-bundle of frames.

Here is a corollary of Theorem 5.1 concerning the complex structure of twistor space.

Corollary 5.1. *If M is a Riemann surface, then $\mathcal{H}^\nabla$ and $\mathcal{V}$ are holomorphic line bundles over $\mathcal{Z}^0_M$.*

Proof. Let $D=\pi^*\nabla-P$ be the connection defined earlier, induced here by the Levi-Civita connection ∇ of $\omega(,J_0)$. First we compute in any dimension

$$\begin{aligned}\mathrm{d}^{\pi^*\nabla}P(X,Y)&=\pi^*\nabla_X(PY)-\pi^*\nabla_Y(PX)-P[X,Y]\\&=\pi^*\nabla_X PY-\pi^*\nabla_Y PX+P(T^D(X,Y)-D_XY+D_YX)\\&=\pi^*\nabla_X PY-\pi^*\nabla_Y PX+P(\pi^*R^\nabla_{X,Y})-\pi^*\nabla_X PY+[PX,PY]\\&\quad+\pi^*\nabla_Y PX-[PY,PX]\\&=P(\pi^*R_{X,Y})+2[PX,PY].\end{aligned}$$

Hence, by a well known formula on the curvature, we have

$$R^D=R^{\pi^*\nabla}-\mathrm{d}^{\pi^*\nabla}P+P\wedge P \tag{5.1}$$

$$=\pi^*R-P(\pi^*R)-P\wedge P. \tag{5.2}$$

Now, recall the twistor space is always a complex two-manifold and D is a $\mathbb{C}$-linear connection. Moreover, in dimension $n=1$ we also have that R^∇ is proportional to ω and so it is type (1,1) for all j in any fibre of the twistor space — an assertion equivalent to π^*R being (1,1) for $\mathcal{J}^\nabla$. On the other hand, $P(\mathcal{J}^{\nabla +}X)=\Phi^+P(X)$ so, if we prove $[\mathfrak{m}_J^+,\mathfrak{m}_J^+]=0$, then we may conclude R^D is type (1,1). The result now follows for both vector bundles referred, by the theorem of Koszul-Malgrange previously mentioned.

If $A,B\in\mathfrak{m}_J$, then

$$J^+AJ^+B=AJ^-J^+B=0$$

where J^+,J^- are the projections onto the + or $-\sqrt{-1}$-eigenbundles. $\square$

As the reader may notice, the result is valid in any dimension once $\mathcal{J}^\nabla$ is integrable. We combine the proof above with Eq. (1.4) in Theorem 1.1.

Finally, we reach a goal: if one assigns a metric to M, then all previous constructions follow and one is left with a new tool in the study of Riemann surfaces. Letting $\mathcal{F}$ denote one of the sheaves of germs of holomorphic sections of $\mathcal{H}^\nabla$ or $\mathcal{V}$, then

$$R^1\pi_*\mathcal{F}$$

may tell us something new about M. Indeed, at the end of the last section we discuss and conjecture that $R^1\pi_*\mathcal{O}$ is non zero.

6. The Penrose Transform

Let Z be a complex manifold of dimension m. Recall that Z is said to be strongly q-pseudoconvex if it admits a smooth exhaustion function which is strongly q-pseudoconvex outside of a compact subset, i.e. there exists $\phi:Z\to\mathbb{R}$ of class C^∞ such that the level sets

$\{x \in Z : \phi(x) < c\}$, $c \in \mathbb{R}$, are relatively compact in Z, the exhaustion, and such that the Levi form

$$L(\phi) : TZ \otimes TZ \longrightarrow \mathbb{R}$$

has at least $m - q + 1$ positive eigenvalues in the complement of a compact subset C. If $C = \emptyset$, then Z is said to be holomorphically q-complete.

Recall that

$$L(\phi) = 4 \sum_{i,j} \frac{\partial^2 \phi}{\partial z_i \partial \overline{z}_j} \mathrm{d}z_i \otimes \mathrm{d}\overline{z}_j$$

is a Hermitian two-tensor, independent of choice of the chart $(z_1, \ldots, z_m)$ of Z. From the definition we see that q-completeness implies $q + 1$-completeness. Holomorphically 1-complete manifolds are known as Stein manifolds.

Theorem 6.1 ([22]). *A simply connected complete Kähler manifold X of everywhere non-positive sectional curvature is a Stein manifold.*

The proof of this theorem due to H. Wu contains the following arguments. Let $d : X \to \mathbb{R}$ be the Riemannian distance function from a fixed point $p \in X$. Then it is proved that d^2 is smooth and strictly plurisubharmonic. It is an exhaustion function due to completeness: a bounded and closed set is compact.

Besides $\mathbb{C}^n$, the canonical example to which the theorem above applies is the Siegel domain. So the square of the distance function in $Sp(2n, \mathbb{R})/U(n)$ with invariant hyperbolic metric is C^∞. Notice the same result does not apply to all components of G/U^l, as their natural metrics may be indefinite. Yet they are Stein spaces as we have remarked earlier.

Now let (M, ω, ∇) be a symplectic manifold of dimension $2n$ with a symplectic connection of Ricci type, i.e. with vanishing Weyl curvature tensor. Consider the twistor space $(\mathcal{Z}^0, \mathcal{J}^\nabla)$, which is then a complex manifold of dimension $n + k$ with $k = n(n + 1)/2 =$ dim Siegel domain G/U^0.

Lemma 6.1. *Let D be a domain in* $\mathbb{C}^m$ *and X a regular complex analytic subspace. If* $\psi \in \mathrm{C}^2_D$ *then*

$$L(\psi)_{|TX \otimes TX} = L(\psi_{|X}).$$

Proof. We know that for every $z \in X$ there is a chart $(z_1, \ldots, z_m)$ in a neighborhood U of z such that $X \cap U = \{z \in U : z_{k+1} = \cdots = z_m = 0\}$. Since $T_z(X \cap U) = \{u \in T_z U : \mathrm{d}z_i(u) = 0, i > k\}$ we find the result just by looking at the definition of the Levi form. $\square$

M always admits a smooth and compatible almost complex structure J, so we define a smooth function h on $\mathcal{Z}^0$ to be the square of the distance in each fibre to the section J — which we know to arise from a smooth Riemannian metric on the vertical bundle ker $\mathrm{d}\pi$.

Theorem 6.2. *If M has a smooth exhaustion function ϕ, then $\mathcal{Z}^0$ is $n+1$-complete.*

Proof. Since it is easy to prove ϕ^2 is also an exhaustion function, we may already assume ϕ to be positive. Now let

$$\psi = h + \phi \circ \pi.$$

This is a smooth and exhaustion function. To prove this notice that h is positive, so the closed level sets of ψ are inside the closed level sets of $\phi \circ \pi$ for the same constant c. These project onto a compact subset K_c of M. Then we have that

$$\begin{aligned}\{j \in \mathcal{Z}^0 : \psi(j) \leq c\} &\subset \{j \in \pi^{-1}(K_c) : \psi(j) \leq c\}\\ &\subset \{j \in \pi^{-1}(K_c) : h(j) \leq c + \sup_{K_c} \phi\}\end{aligned}$$

and, since the biggest set is compact, the closed level sets of ψ are compact.

Now, for any $x \in M$, we apply the lemma to the complex submanifold $\pi^{-1}(x)$ and use the previous theorem to find that $L(\psi_{|\pi^{-1}(x)}) = L(h_{|\pi^{-1}(x)})$ has k positive eigenvalues. Since $L(\psi)$ is Hermitian symmetric, there is an orthogonal complement for $\ker \mathrm{d}\pi$ and we may conclude that $L(\psi)$ has at least k positive eigenvalues. Hence $\mathcal{Z}^0$ is q-complete, where q is such that $n + k - q + 1 = k$. $\square$

Example 1. If M is compact we may take $\phi = 0$ in the theorem above. In particular, $\mathcal{Z}^0_{\mathbb{CP}^n}$ is $n+1$-complete (and not less).

Example 2. If M has some Riemannian structure for which there is a pole, i.e. there exists $x_0 \in M$ such that $\exp : T_{x_0}M \to M$ is a diffeomorphism, then we may take $\phi = ||\exp^{-1}||^2$.

Example 3. Let $M = B_\epsilon(0)$, the open ball of radius ϵ in $(\mathbb{R}^{2n}, \omega_0)$. For this case we find $\phi(x) = -\log(\epsilon^2 - ||x||^2)$, which is the famous function of K. Oka.

One must realise now that the difficult thing is to find completeness below $n+1$. This will certainly involve the horizontal part of $\mathcal{J}^\nabla$, which so much characterises twistor spaces.

Remark. In general, it is impossible to find a better result than that of the theorem: we know the Levi-Civita connection of $\mathbb{CP}^n$ is of Ricci type and we have seen that parallel complex structures embed holomorphically into the twistor space. On the other hand it is known that a q-complete space does not have n-dimensional compact analytic submanifolds, for any $n \geq q$.

By the same token, the *Kählerian* twistor space $\mathcal{Z}_{\mathbb{T}^{2n}}$ is just holomorphically $n+1$-complete, and no less.

However, with some restriction, it may well happen that it is possible to carry on. As we have seen in Example 1.1 of the Examples section, together with Theorem 4.1, the twistor space of $\mathbb{R}^2$ with trivial connection ∇^0, and hence with all $\sigma\cdot\nabla^0$, is 1-complete or Stein (such property is preserved under biholomorphism).

We are now ready to show the Penrose Transform.

In a parallelism with what was done in [3,9,17], in the celebrated Riemannian case of $\mathbb{CP}^3 \to S^4$, we define the "Penrose Transform" in the symplectic context to be the direct image of any complex analytic sheaf over twistor space onto the base manifold. Thus a functor $\mathcal{O} \to \mathrm{C}^\infty$.

Theorem 6.3. *Let* (M, ω, ∇) *be as above and* $\mathcal{F}$ *a coherent analytic sheaf over* $\mathcal{Z}^0$. *Then*

$$R^q\pi_*\mathcal{F} = 0, \quad \forall q \geq n+1.$$

Proof. Recall $R^q\pi_*\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H^q(\pi^{-1}U, \mathcal{F})$. Hence the stalk at $x \in M$ is

$$\lim_{U\ni x} \operatorname{ind} H^q(\pi^{-1}U, \mathcal{F}).$$

Now, for a sufficiently small neighborhood U of x, there is a chart $\sigma : U \to B \subset \mathbb{R}^{2n}$ such that $\sigma^*\omega_0 = \omega$ and $\sigma(x) = 0$. Since we have a theorem saying there is a biholomorphism

$$\Sigma : (\mathcal{Z}^0_U, \mathcal{J}^\nabla) \longrightarrow (\mathcal{Z}^0_B, \mathcal{J}^{\sigma\cdot\nabla}),$$

we may suppose our base space is B and the *coherent* analytic sheaf is $\Sigma_*\mathcal{F}$.

Finally, the $\{B_\epsilon(0)\}_{\epsilon>0}$ form a basis for the neighborhoods of 0 and, by example 3 above, all $\mathcal{Z}^0_{B_\epsilon} = \pi^{-1}(B_\epsilon)$ are n+1-complete. By definition of inductive limit we find that $(R^q\pi_*\mathcal{F})_x = 0, \forall q \geq n+1$, appealing to Andreotti-Grauert's "t. de finitude pour la cohomologie des espaces complexes" (cf. [2]). □

Although we know $H^q(\pi^{-1}(x), \iota^*\mathcal{F}) = 0, \forall q \geq 1$, where ι is the inclusion map, one has to notice in the above proof that the $\{\pi^{-1}(U)\}$ do not form a basis of the neighborhoods of $\pi^{-1}(x)$, as they always do in the Riemannian case (the fibre is compact).

Remark. Consider Example 1.1 in Section 3. Recall the global chart $(z, w) \mapsto (\xi, w)$ where $\xi = z - w\bar{z}$ is a complex coordinate. Let us fix $z \in \mathbb{R}^2$ and denote

$$X_\varepsilon = \{(\xi, w) \in \mathbb{C}^2 : |w| < 1,\ |z| < \varepsilon\}$$

i.e. the image in $\mathbb{C}^2$ of $\pi^{-1}(B_\varepsilon(z))$. The condition $|z| < \varepsilon$, where $z = \frac{\xi + w\bar{\xi}}{1-|w|^2}$, shows that X_ε is not pseudoconvex: we can show that some regions in the boundary are convex and others are concave. Pseudoconvexity in $\mathbb{C}^n$, $n > 1$, is the same as being Stein, so we may follow [16] and conjecture that $H^1(X_\varepsilon, \mathcal{O}) \neq 0$.

If such conjecture is true, then we can also deduce that our sheaves $R^1\pi_*\mathcal{F}$ existing always over Riemann surfaces (see Section 5.2) are not in general completely trivial.

References

[1] R. Albuquerque, On the twistor theory of symplectic manifolds, PhD thesis, University of Warwick, 2002.
[2] A. Andreotti, H. Grauert, Théorème de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France 90 (1962) 193–259.
[3] M. Atiyah, Geometry of Yang-Mills Fields, Scuola Normale Superiore, Pisa, 1979.
[4] M. Atiyah, N. Hitchin, I. Singer, Self-duality in four-dimensional riemannian geometry, Proc. Roy. Soc. Lond., A 362 (1711) (1978) 425–461.
[5] F. Bourgeois, M. Cahen, A variational principle for symplectic connections, J. Geom. Phys. 3 (2000) 233–265.
[6] M. Cahen, S. Gutt, J. Horowitz, J. Rawnsley, Homogeneous symplectic manifolds with ricci-type curvature, J. Geom. Phys. 2 (2001) 140–151.
[7] M. Cahen, S. Gutt, J. Horowitz, J. Rawnsley, Moduli Space of Symplectic Connections of Ricci Type on t^{2n}; A Formal Approach, Warwick Preprint, 2002.
[8] M. Cahen, S. Gutt, J. Rawnsley, Symmetric symplectic spaces with Ricci-type curvature, Conférence Moshé Flato 1999, vol. II, Dijon, Kluwer Acad., Math. Phys. Stud., 22, 2000.
[9] A. Douady, J.-L. Verdier (Eds.), Les équations de Yang-Mills, Astérisque, Séminaire E.N.S., 1977–1978, Soc. Math. de France, Paris, 1980, pp. 71–72.
[10] N.J. Hitchin, Kählerian twistor spaces, Proc. Lond. Math. Soc. (3) 43 (1) (1981) 133–150.
[11] S. Kobayashi, Differential geometry of complex vector bundles, Iwanami Shoten, Princeton University Press, 1987.
[12] D. McDuff, D. Salamon, Introduction to symplectic topology, Oxford Mathematical Monographs, Oxford University Press, New York, 1998.
[13] A. Nannicini, Twistor bundles of almost symplectic manifolds, Rend. Istit. Mat. Univ. Trieste 30 (1998) 91–106.
[14] A. Nannicini, Twistor methods in conformal almost symplectic geometry, Rend. Istit. Mat. Univ. Trieste 34 (2002) 215–234.
[15] N. O'Brian, J. Rawnsley, Twistor spaces, Ann. Global Anal. Geometry 3 (1) (1985) 29–58.
[16] T. Peternell, Cohomology, Several Complex Variables VII, vol. 74, Encyclopedia of Mathematical Sciience, Springer, 1990.
[17] J. Rawnsley, On the atiyah-hitchin-drinfel'd-manin vanishing theorem for cohomology groups of instanton bundles, Math. Ann. 241 241 (1) (1979) 43–56.
[18] J. Rawnsley, *f*-Structures, *f*-twistor spaces and harmonic maps, Geometry Seminar "Luigi Bianchi" II, Springer, Berlin, 1985, pp. 85–159.
[19] S. Salamon, Harmonic and holomorphic maps. Springer, Berlin, pages 161–224, 1985. Geometry seminar "Luigi Bianchi" II.
[20] I. Vaisman, Symplectic curvature tensors, Monatsh. Math. 100 (4) (1985) 299–327.
[21] I. Vaisman, Symplectic twistor spaces, J. Geom. Phys. 3 (4) (1986) 507–524.
[22] H. Wu, An elementary method in the study of nonnegative curvature, Acta Math. 142 (1–2) (1979) 57–78.